

$E_{7(7)}$ Symmetry and Finiteness of $\mathcal{N} = 8$ Supergravity

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Abstract

We study $\mathcal{N} = 8$ supergravity deformed by the presence of the candidate counterterms. We show that even though they are invariant under undeformed $E_{7(7)}$, all of the candidate counterterms violate the deformed $E_{7(7)}$ current conservation. The same conclusion follows from the uniqueness of the Lorentz and $SU(8)$ covariant, $E_{7(7)}$ invariant unitarity constraint expressing the 56-dimensional $E_{7(7)}$ doublet via 28 independent vectors, in agreement with the $\frac{E_{7(7)}}{SU(8)}$ coset space geometry. Therefore $E_{7(7)}$ duality predicts the all-loop UV finiteness of perturbative $\mathcal{N} = 8$ supergravity.

1 Introduction

Recently we argued in [1] that $\mathcal{N} = 8$ supergravity [2] is perturbatively UV finite. The proof was rather complicated, based on a relation between the real and chiral versions of the off-shell light-cone superspace. It generalized the perturbative supergraph non-renormalization theorem for the superpotential. It was necessary to compare properties of the light-cone superspace with that of the covariant superspace and combine it all with the recent approach based on the helicity amplitude computations.

In this paper we will study this issue using a simpler set of more familiar tools, which do not require knowledge of the off-shell light-cone superspace and helicity amplitudes. We will analyze all candidate counterterms [3, 4] using the standard Lorentz covariant on shell superspace approach [5]. We will show that the power of the continuous global $E_{7(7)}(\mathbb{R})$ duality symmetry and of the corresponding current conservation was underestimated. It is, in fact, strong enough to forbid all perturbative¹ UV divergences which are consistent with the gauge symmetries of the theory.

The discovery of the 3-loop UV finiteness [7] of $\mathcal{N} = 8$ supergravity attracted attention to a possibility that the hidden $E_{7(7)}$ symmetry of the theory may be relevant to the UV properties of the theory [8–11]. The full set of non-linear $E_{7(7)}$ invariant counterterms was constructed long time ago in [3, 4], based on $\mathcal{N} = 8$ supergravity in the on shell superspace [5]. Recently the explanation of the 3-loop UV finiteness due to unbroken $E_{7(7)}$ continuous symmetry was proposed in [12]. $E_{7(7)}$ constraints on linearized counterterms in $\mathcal{N} = 8$ supergravity were studied in [13]. A recent review of the candidate counterterms from the amplitude viewpoint is available in [14].

Thus, it was recognized that $E_{7(7)}$ symmetry may impose stringent constraints on the structure of the candidate counterterms. However, the standard lore was that as long as the candidate counterterms are $E_{7(7)}$ invariant, they should be allowed by the $E_{7(7)}$ symmetry of the theory. As we are going to show, this is not the case; adding $E_{7(7)}$ invariant counterterms to the theory may break the $E_{7(7)}$ symmetry of the theory deformed by the addition of these counterterms.

The unusual properties of continuous global duality symmetries make the standard Noether construction not useful for the vector field part of the symmetry. To study duality symmetries one has to use the Noether-Gaillard-Zumino type construction [15, 16], which provides the conserved current for such symmetries and guaranties that the equations of motion are duality invariant.² In application to

¹The $E_{7(7)}(\mathbb{R})$ symmetry is expected to be a symmetry of perturbative $\mathcal{N} = 8$ supergravity. It is broken to an arithmetic subgroup $E_{7(7)}(\mathbb{Z})$ by non-perturbative effects. However, one can still use $E_{7(7)}(\mathbb{R})$ symmetry for investigation of UV divergences in the perturbation theory; see [6] for the recent discussion of this issue.

²The *bona fide* Noether current of the Lorentz non-covariant version of the theory developed in [17] and the Hamiltonian approach in [18] may be used for an alternative analysis of the UV properties of $\mathcal{N} = 8$ supergravity.

$\mathcal{N} = 8$ supergravity, the current of the classical theory was constructed explicitly in [9] by embedding $E_{7(7)}$ into $Sp(2n, \mathbb{R})$. In this paper we derive a particularly useful form of the identity equivalent to $E_{7(7)}$ current conservation.

The new developments suggest that we may need to reconsider the predictions of $E_{7(7)}$. The “counterterm wisdom” was that the counterterms should preserve the symmetries of the classical action. Also, the counterterms which vanish on shell may be removable by a gauge choice. For example, the counterterms in pure gravity depend on the Riemann-Christoffel tensor $R_{\mu\nu\lambda\delta}$ and its covariant derivatives. The 2-loop counterterm of the form $R_{\mu\nu\lambda\delta} R^{\lambda\delta\kappa\eta} R_{\kappa\eta}{}^{\mu\nu}$ can be added to the classical Lagrangian without affecting the symmetries of the theory: the Lagrangian with the counterterms

$$S_{\text{deformed}} = S_0 + S_{\text{CT}} = \int \sqrt{-g} \left(\frac{1}{2\kappa^2} R + a\kappa^2 R_{\mu\nu\lambda\delta} R^{\lambda\delta\kappa\eta} R_{\kappa\eta}{}^{\mu\nu} + \dots \right) \quad (1.1)$$

remains invariant under the standard, undeformed general covariance transformations

$$(\Delta g_{\mu\nu})_0 = (\Delta g_{\mu\nu})_{\text{deformed}} = D_\mu \xi_\nu(x) + D_\nu \xi_\mu(x) , \quad (1.2)$$

and is ready to absorb the UV divergences. Note that the presence of the counterterms in the deformed action does not require the deformation of the symmetry! The situation with other gauge symmetries is very similar. When the classical action is deformed by counterterms,

$$S_{\text{deformed}} = S_0 + S_{\text{CT}} , \quad (1.3)$$

it remains invariant under the undeformed gauge symmetries $\Delta_0^{\text{gauge}} S_{\text{deformed}} = \Delta_0^{\text{gauge}} (S_0 + S_{\text{CT}}) = 0$. *The situation with continuous global duality symmetries is more delicate, namely, even the classical Lagrangian is not invariant under duality symmetry, only equations of motion are.*³ The current consists of the standard Noether contribution for all fields but vectors, $j_\mu = \frac{\partial L}{\partial \varphi_{,\mu}} \Delta \varphi$, and of the Gaillard-Zumino current for vector fields, \hat{J}_μ , so that the total current is conserved,

$$\partial_\mu J_{\text{NGZ}}^\mu = \partial_\mu j_\mu + \partial_\mu \hat{J}_\mu = 0 . \quad (1.4)$$

The fact that the Lagrangian is not invariant under duality leads to a dramatic difference with regard to the properties of the admissible counterterms, assuming that the corresponding duality symmetry has no anomalies. The difference comes from the properties of the transformations of the deformed Lagrangian including the classical part and the counterterms. In addition to the field strength $F = dA$ present in the action, the symmetry also involves the dual field strength defined by the variation of the

³ The prototype of electric-magnetic duality is the free Maxwell theory: in vacuum equations of motion and Bianchi identities are invariant under rotations. The Maxwell Lagrangian, $E^2 - B^2$, is not invariant under $E \leftrightarrow B$ rotation, only the Hamiltonian, $E^2 + B^2$, is invariant, and the Noether-Gaillard-Zumino current is conserved.

action over F , namely $\tilde{G} = 2\frac{\delta S}{\delta F}$. This is a basic unitarity issue. For example, in $\mathcal{N} = 8$ supergravity there are 28 real fields, but the $E_{7(7)}$ symmetry, mixing 28 Bianchi identities and 28 field equations of motion, requires a 56-dimensional doublet of fields (F, G) , where 28 G 's depend on 28 F 's and on scalars. The symmetry is

$$\Delta\tilde{F} = A\tilde{F} + B\tilde{G} = A\tilde{F} + 2B\frac{\delta(S_0 + S_{\text{CT}})}{\delta F} = \Delta_0\tilde{F} + B\frac{\delta S_{\text{CT}}}{\delta F}, \quad (1.5)$$

where A, B are infinitesimal parameters of transformation. The $\mathcal{N} = 8$ supergravity counterterms constructed in [3, 4] are $E_{7(7)}$ invariant under the undeformed symmetry Δ_0 associated with the classical action. The counterterms depend on F , therefore $\frac{\delta S_{\text{CT}}}{\delta F} \neq 0$, which deforms $E_{7(7)}$ symmetry. In $\mathcal{N} = 8$ supergravity $B = \text{Im}\Lambda + \text{Im}\Sigma$ are the off diagonal components of duality symmetries, which mix Bianchi identities with equations of motion:

$$\Delta\partial_\mu\tilde{F}^{\mu\nu} = B\partial_\mu\tilde{G}^{\mu\nu}. \quad (1.6)$$

Therefore the Lagrangian including the counterterms does not automatically lead to the $E_{7(7)}$ symmetric deformed equations of motion. We have to find out whether the $E_{7(7)}$ symmetry deformed by the counterterms remains valid. This means that we have to find out whether equations of motion/Bianchi identities transform into each other by the deformed $E_{7(7)}$ transformations, and in this way to find the implications of the continuous global non-compact $E_{7(7)}$ symmetry for the quantum theory.

In Sec. 2 we introduce the concept of a duality doublet (F, G) which has a double amount of field strengths as compared to the one present in the action: G is a functional of F and other fields, or vice versa. We use the example of classical $\mathcal{N} > 2$ supergravity to explain it. In Sec. 3 we discuss duality symmetry and Noether-Gaillard-Zumino identity [15, 16], for any symmetry which may be embedded into $Sp(2n, \mathbb{R})$. We derive a particular form of the duality identity for the case of $\mathcal{N} = 8$ supergravity, where the counterterms are invariant under undeformed duality. In Sec. 4 we analyze the 3-loop and higher loop counterterms of $\mathcal{N} = 8$ supergravity, compute the deformation of the dual field strength caused by the counterterms and show that the deformation breaks $E_{7(7)}$ current conservation identity. In Sec. 5 we argue that the unitarity constraint which allows to express half of the 56-dimensional $E_{7(7)}$ doublet via the independent 28 fields is unique. This leads to an independent argument that the hidden $E_{7(7)}$ symmetry combined with all manifest gauge symmetries of $\mathcal{N} = 8$ supergravity forbids all counterterms in $\mathcal{N} = 8$ supergravity. In Sec. 6 we discuss our findings and provide a short technical summary of the results. Appendix A contains a derivation of the NGZ identity and simple expressions for the scalar and vector part of the $E_{7(7)}$ current of the $SU(8)$ locally invariant $\mathcal{N} = 8$ supergravity. In Appendix B we discuss the Hamiltonian approach to dualities.

2 Duality doublet (F, G)

2.1 Classical $\mathcal{N} > 2$ supergravity

Here we start with the classical $d = 4$ $\mathcal{N} > 2$ pure supergravity and explain its duality symmetry.⁴ In absence of fermions four-dimensional supergravities depend on metric, vectors and scalars. In particular, the action depends on Abelian vectors \mathcal{A}_μ^Λ via the field strength $F_{\mu\nu}^\Lambda = \partial_\mu \mathcal{A}_\nu^\Lambda - \partial_\nu \mathcal{A}_\mu^\Lambda$, on scalars and on metric

$$S_{\text{cl}}(F, \phi, g) = \frac{1}{4\kappa^2} \int d^4x e \left(-\frac{1}{2}R + \text{Im}\mathcal{N}_{\Lambda\Sigma} F_{\mu\nu}^\Lambda F^{\mu\nu\Sigma} + \frac{1}{2e} \text{Re}\mathcal{N}_{\Lambda\Sigma} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^\Lambda F_{\rho\sigma}^\Sigma + \frac{1}{2} g_{ij}(\phi) \partial_\mu \phi^i \partial_\mu \phi^j \right). \quad (2.1)$$

Here the kinetic terms for vectors $\mathcal{N}_{\Lambda\Sigma}(\phi)$ depends on scalars. The manifold of scalars is a coset space G/H . In case of $\mathcal{N} = 8$ supergravity $G/H = E_{7(7)}/SU(8)$. There is a Bianchi identity for $F_{\mu\nu}^\Lambda = \partial_\mu \mathcal{A}_\nu^\Lambda - \partial_\nu \mathcal{A}_\mu^\Lambda$,

$$\partial_\mu \tilde{F}^{\mu\nu\Lambda} = 0. \quad (2.2)$$

The dual field strength $G_\Lambda^{\mu\nu}$ is defined as a derivative of the action over $F_{\mu\nu}$, namely $\tilde{G}_\Lambda^{\mu\nu} = 2 \frac{\delta S(F, \phi, g)}{\delta F_{\mu\nu}^\Lambda}$. Equations of motion provide the Bianchi identity for the dual field strength

$$\partial_\mu \frac{\delta S}{\delta F_{\mu\nu}^\Lambda} = 0 \quad \Rightarrow \quad \partial_\mu \tilde{G}^{\mu\nu} = 0 \quad \Rightarrow \quad G_{\mu\nu} = \partial_\mu \mathcal{B}_\nu - \partial_\nu \mathcal{B}_\mu. \quad (2.3)$$

The dual potential \mathcal{B}_μ exists only when the equation of motion of the deformed theory are satisfied. One can evaluate G using the action in (2.1) and one finds in absence of fermions⁵

$$\tilde{G}_\Lambda^{\mu\nu} = 2 \frac{\delta S}{\delta F_{\mu\nu}^\Lambda} = \frac{1}{\kappa^2} \left(e \text{Im}\mathcal{N}_{\Lambda\Sigma} F^{\mu\nu\Sigma} + \frac{1}{2} \text{Re}\mathcal{N}_{\Lambda\Sigma} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}^\Sigma \right) = \tilde{G}_\Lambda^{\mu\nu}(F, \phi, g). \quad (2.4)$$

For $\mathcal{N} > 2$ supergravity duality symmetry can be embedded into $Sp(2n, \mathbb{R})$. It requires that the vector doublet $(F, G = 2 \frac{\delta S}{\delta F})$ transforms in the fundamental of $Sp(2n, \mathbb{R})$

$$\begin{pmatrix} F' \\ G' \end{pmatrix} = \mathcal{S} \begin{pmatrix} F \\ G \end{pmatrix}, \quad \mathcal{S} \equiv \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix}. \quad (2.5)$$

Here the matrix \mathcal{S} is symplectic⁶, it has real elements that satisfy the following conditions: $\hat{A}^\top \hat{C} - \hat{C}^\top \hat{A} = \hat{B}^\top \hat{D} - \hat{D}^\top \hat{B} = 0$ and $\hat{A}^\top \hat{D} - \hat{C}^\top \hat{B} = 1$. The gauge kinetic term \mathcal{N} transforms via fractional

⁴We use notation of [16], including $\tilde{G}^{\mu\nu} = \sqrt{g} G^{*\mu\nu}$ where the Hodge dual of a two form is $\Omega_{\mu\nu}^* = \frac{1}{2} \sqrt{g} \epsilon_{\mu\nu\lambda\sigma} \Omega^{\lambda\sigma}$. In the flat case $\tilde{G}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\sigma} G_{\lambda\sigma}$ and $G_{\mu\nu} = -\frac{1}{2} \epsilon_{\mu\nu\lambda\sigma} \tilde{G}^{\lambda\sigma}$ ($\epsilon^{0123} = -\epsilon_{0123} = 1$). We will also suppress spacetime indices so that for example $F\tilde{G} = F_{\mu\nu} \tilde{G}^{\mu\nu}$. It is useful to remember that $F\tilde{G} = \tilde{F}G$ and $\tilde{\tilde{F}} = -F$, $\tilde{F}\tilde{G} = -FG$ where $FG = F^{\mu\nu} G_{\mu\nu}$.

⁵When fermions are present in classical supergravity, \tilde{G} has also terms independent on F .

⁶We will later use the infinitesimal form of duality symmetry. This will correspond to $\hat{A} \approx 1 + A$, $\hat{B} \approx B$, $\hat{C} \approx C$, $\hat{D} \approx 1 + D$.

transformations

$$\mathcal{N}' = (\hat{C} + \hat{D}\mathcal{N})(\hat{A} + \hat{B}\mathcal{N})^{-1} . \quad (2.6)$$

Note that the duality (2.5) requires that

$$G' = \hat{C}F + \hat{D}G \quad \Rightarrow \quad \tilde{G}' = \hat{C}\tilde{F} + \hat{D}\tilde{G} . \quad (2.7)$$

On the other hand one can evaluate G' using its expression in (2.4) where it depends on F and scalars and using that from (2.5) $F' = \hat{A}F + \hat{B}G$, as well as the transformations of scalars in (2.6). This means that the transformation of the dual field strength $G(F, \phi, g)$ computed via its relation to F, ϕ, g using the chain rule, gives the same answer as the one required by the symmetry in the form (2.5). The fact that classical supergravities have duality symmetries with the conserved current means that \tilde{G}' in (2.7) coincides with the expression following from the constraint between G and F, ϕ in (2.4). This is the essence of Noether-Gaillard-Zumino $Sp(2n, \mathbb{R})$ construction for duality invariant theories.

When counterterms are added to the action $F_{\mu\nu}^\Lambda$, of course, remains undeformed, $F_{\mu\nu}^\Lambda = \partial_\mu \mathcal{A}_\nu^\Lambda - \partial_\nu \mathcal{A}_\mu^\Lambda$. However the dual field strength $G_{\mu\nu}$ is deformed when the action is deformed

$$\tilde{G}^{\mu\nu \text{ deformed}}(F, \phi, g) = 2 \frac{\delta S_0}{\delta F_{\mu\nu}}(F, \phi, g) + 2 \frac{\delta S_{\text{CT}}}{\delta F_{\mu\nu}}(F, \phi, g) . \quad (2.8)$$

The problem we will address with regard to a deformed action $S_0 + S_{\text{CT}}$ is: given the explicit dependence of $\tilde{G}^{\mu\nu \text{ deformed}}$ on (F, ϕ, g) , which we will compute from the S_{CT} , will it provide the deformed duality transformation corresponding to the doublet transformations of (F, G^{deformed}) ?

3 Noether-Gaillard-Zumino (NGZ) $Sp(2n, \mathbb{R})$ duality identity

Here we review the NGZ construction [15, 16] for theories with $Sp(2n, \mathbb{R})$ duality, which has some real vector fields \mathcal{A}_μ^Λ and other fields φ^α , which include scalars, spinors, metric. There is an infinitesimal $Sp(2n, \mathbb{R})$ transformation, which acts on $Sp(2n, \mathbb{R})$ doublet of vectors field strength (F, G) as follows:

$$\Delta \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} , \quad \Delta \begin{pmatrix} \tilde{F} \\ \tilde{G} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \tilde{F} \\ \tilde{G} \end{pmatrix} . \quad (3.1)$$

$$A^\top = -D , \quad B^\top = B , \quad C^\top = C . \quad (3.2)$$

Here A, B, C, D are the infinitesimal global (space-time independent) parameters of the transformations, arbitrary real $n \times n$ matrices satisfying (3.2). It is an infinitesimal version of (2.5). The upper component of the doublet $F_{\mu\nu}^\Lambda = \partial_\mu \mathcal{A}_\nu^\Lambda - \partial_\nu \mathcal{A}_\mu^\Lambda$ is the field strength of the vector field, the down component is a dual field strength,

$$\tilde{G}_\Lambda^{\mu\nu}[F, \varphi] \equiv 2 \frac{\delta S[F, \varphi]}{\delta F_{\mu\nu}^\Lambda} . \quad (3.3)$$

Duality symmetry on φ^α fields is of the form

$$\Delta\varphi^\alpha = \Xi^\alpha(\varphi) . \quad (3.4)$$

There is a consistency requirement here that the dual field strength transforms according to (3.1) using the chain rule, when expressed as a functional of F and φ . This consistency condition⁷ is given in the form of Noether-Gaillard-Zumino $Sp(2n, \mathbb{R})$ duality identity which we will present and use below. If the identity is satisfied, duality symmetry of equations of motion follows from (3.1) since it mixes Bianchi identities with the equations of motion:

$$\Delta \begin{pmatrix} \partial_\mu \tilde{F}^{\mu\nu} \\ \partial_\mu \tilde{G}^{\mu\nu} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \partial_\mu \tilde{F}^{\mu\nu} \\ \partial_\mu \tilde{G}^{\mu\nu} \end{pmatrix} . \quad (3.5)$$

A consistency of the $Sp(2n, \mathbb{R})$ duality symmetry requires that *the Lagrangian must transform under duality in a certain way, defined by NGZ identity* [15, 16]

$$\boxed{\frac{\delta}{\delta F^\Lambda} \left(S[F', \varphi'] - S[F, \varphi] - \frac{1}{4} \int (\tilde{F} C F + \tilde{G} B G) \right) = 0.} \quad (3.6)$$

This identity is suitable for the situation that the action depends on $F_{\mu\nu}$ and on its derivatives. Instead of partial Lagrangian derivatives $\frac{\partial L}{\partial F^\Lambda}$ one has to use the variational derivatives of the action, $\frac{\delta S}{\delta F^\Lambda}$, as suggested in (3.6). We present a derivation of the identity (3.6) in Appendix A.

3.1 Using the symmetry of counterterms under undeformed duality

All deformations will be denoted by the hat symbol.

$$S = S_0 + \hat{S} , \quad G = G_0 + \hat{G} , \quad \tilde{G} = \tilde{G}_0 + \hat{\tilde{G}} , \quad \Delta = \Delta_0 + \hat{\Delta} , \quad (3.7)$$

F is not deformed, $\Delta\varphi^\alpha$ is not deformed.

$$\Delta S = \Delta_0 S_0 + \Delta_0 \hat{S} + \hat{\Delta} S_0 + \hat{\Delta} \hat{S} . \quad (3.8)$$

If the counterterms are invariant under undeformed duality, like in $\mathcal{N} = 8$ supergravity, it means that

$$\Delta_0 S_{\text{CT}} = \Delta_0 \hat{S} = 0 . \quad (3.9)$$

The NGZ identity (3.6) takes the form

$$\frac{\delta}{\delta F^\Lambda} \left(\Delta_0 S_0 + \hat{\Delta} S_0 + \hat{\Delta} \hat{S} - \frac{1}{4} \int (\tilde{F} C F + \tilde{G} B G) \right) = 0 . \quad (3.10)$$

⁷ It is equivalent to the Noether current conservation. The current has some unusual properties, see Appendix B.

The undeformed symmetry of the undeformed action cancels in this expression since

$$\frac{\delta}{\delta F^\Lambda} \left(\Delta_0 S_0 - \frac{1}{4} \int (\tilde{F} C F + \tilde{G}_0 B G_0) \right) = 0 . \quad (3.11)$$

and the remaining identity is (taking into account that $GB\tilde{G} = (G_0 + \hat{G})B(\tilde{G}_0 + \hat{\tilde{G}})$)

$$\frac{\delta}{\delta F^\Lambda} \left(\hat{\Delta} S - \frac{1}{4} \int (2\tilde{G}_0 B \hat{G} + \hat{\tilde{G}} B \hat{G}) \right) = 0 . \quad (3.12)$$

Note that the deformation of duality enters only via the deformation of the symmetry on vectors

$$\hat{\Delta} S = \frac{\delta S}{\delta F} B \hat{G} = \frac{1}{2} \tilde{G} B \hat{G} = \frac{1}{2} \tilde{G}_0 B \hat{G} + \frac{1}{2} \hat{\tilde{G}} B \hat{G} . \quad (3.13)$$

We plug this back to NGZ identity (3.12) and find that

$$\frac{\delta}{\delta F^\Lambda} \int \left(\frac{1}{2} \tilde{G}_0 B \hat{G} + \frac{1}{2} \hat{\tilde{G}} B \hat{G} - \frac{1}{4} (2\tilde{G}_0 B \hat{G} + \hat{\tilde{G}} B \hat{G}) \right) = 0 , \quad (3.14)$$

which requires that

$$\boxed{\frac{\delta}{\delta F^\Lambda} \int (\hat{\tilde{G}} B \hat{G}) = 0} . \quad (3.15)$$

This puts a strong restriction on the deformation of the action \hat{S} and the possible form of the deformation of the dual field strength $\hat{\tilde{G}}$.

4 $\mathcal{N} = 8$ Supergravity Counterterms and $E_{7(7)}$ Identity

The on shell superspace [5] provides a geometric construction of counterterms [3, 4] where the torsion and curvature superspace tensors T_{KL}^M and R_{MNKL} are manifestly Lorentz and $SU(8)$ covariant.

$$L_{CT} = L_{CT} \left(T_{KL}^P(x, \theta), R_{PQKL}(x, \theta) \right) . \quad (4.1)$$

These tangent space tensors transforms as tensors under the local Lorentz and local $SU(8)$ transformations and they are neutral under the classical, undeformed $E_{7(7)}$, it is hidden. On shell means that every superfield satisfies a non-linear classical equation of motion. The counterterms depend on $E_{7(7)}$ non-covariant field strength $F_{\mu\nu}^{IJ}$ and on scalars \mathcal{V} , but only in a combination which does not transform on undeformed $E_{7(7)}$.

1. *Vectors*: The $E_{7(7)}$ vector doublet in classical on shell $\mathcal{N} = 8$ supergravity is defined as follows:

$$d(\mathcal{X}_{IJ}^0, \bar{\mathcal{X}}^{0IJ}) = 0 , \quad (4.2)$$

which means that on shell there 28 complex or 56 real potentials: 28 \mathcal{A}^{IJ} in the classical action and another 28 \mathcal{B}_{IJ}^0 are dual

$$\mathcal{X}_{IJ}^0 = d(\mathcal{B}_{IJ}^0 + i\mathcal{A}^{IJ}) , \quad \bar{\mathcal{X}}^{0IJ} = d(\mathcal{B}_{IJ}^0 - i\mathcal{A}^{IJ}) . \quad (4.3)$$

Here $\mathcal{A}^{IJ} = dx^\mu \mathcal{A}_\mu^{IJ}$ and $\mathcal{B}_{IJ} = dx^\mu \mathcal{B}_{\mu IJ}$ are real and

$$F_{\mu\nu}^{IJ} = \partial_\mu \mathcal{A}_\nu^{IJ} - \partial_\nu \mathcal{A}_\mu^{IJ}, \quad G_{\mu\nu IJ}^0 = \partial_\mu \mathcal{B}_{\nu IJ}^0 - \partial_\nu \mathcal{B}_{\mu IJ}^0 = 2 \frac{\partial S_0[F, \phi]}{\partial F_{\mu\nu}^{IJ}}. \quad (4.4)$$

The spin-one field strengths which transforms as a doublet under $SU(8)$ depend on the $U(1)$ field strength $E_{7(7)}$ doublet $(\mathcal{X}_{IJ}^0, \bar{\mathcal{X}}^{0IJ})$ and on scalars \mathcal{V} as follows

$$(\mathcal{F}_{ij}^0, \bar{\mathcal{F}}^{0ij}) = (\mathcal{X}_{IJ}^0, \bar{\mathcal{X}}^{0IJ}) \mathcal{V}. \quad (4.5)$$

Here ij are the $SU(8)$ and IJ are the $E_{7(7)}$ indices and

$$\mathcal{V} = \begin{pmatrix} U^{IJ}{}_{ij} & \bar{V}^{IJij} \\ V_{IJij} & \bar{U}_{IJ}{}^{ij} \end{pmatrix}. \quad (4.6)$$

is a vielbein-like object describing the 133 scalars of $\mathcal{N} = 8$ supergravity with an unbroken local $SU(8)$. The corresponding coset space geometry is $\frac{E_{7(7)}}{SU(8)}$. When local $SU(8)$ symmetry is gauge-fixed, for example in the unitary gauge, $\mathcal{V} = \mathcal{V}^\dagger$, only 70 physical scalars remain: $\phi_{ijkl} = \frac{1}{4!} \epsilon_{ijklmnpq} \bar{\phi}^{mnpq}$. There is no difference between IJ and ij indices anymore and only global $SU(8)$ remains as the symmetry of the action, whereas the global $E_{7(7)}$ is the symmetry mixing equations of motion with Bianchi identities. The non-linear $E_{7(7)}$ symmetry acts on all (but metric) fields of the theory in $\mathcal{V} = \mathcal{V}^\dagger$ gauge, the compact form of the transformations is given in [9].

The counterterms in [3, 4] are known in the form of the theory with local not gauge-fixed $SU(8)$ symmetry. This simplifies the analysis of duality, where \mathcal{V} is a 133-component group element of $E_{7(7)}$ which transforms under local $SU(8)$ and by a global $E_{7(7)}$ symmetry: the global $E_{7(7)}$ acts on the capital indices and the local $SU(8)$ acts on the lower case indices

$$\mathcal{V}' = E^{-1} \mathcal{V} U(x), \quad (\mathcal{X}_{IJ}^0, \bar{\mathcal{X}}^{0IJ})' = (\mathcal{X}_{IJ}^0, \bar{\mathcal{X}}^{0IJ}) E. \quad (4.7)$$

$$E = e^{G_{E_{7(7)}}}, \quad G_{E_{7(7)}} = \begin{pmatrix} 2\delta_{[K}^{[I} \Lambda^{J]}{}_{L]} & \bar{\Sigma}^{IJKL} \\ \Sigma_{IJKL} & 2\delta_{[I}^{[K} \Lambda^{L]}{}_{J]} \end{pmatrix}, \quad (4.8)$$

$$U(x) = \exp G_{SU(8)}, \quad G_{SU(8)}(x) = \begin{pmatrix} \delta_{[k}^{[i} \Lambda^{j]}{}_{l]}(x) & 0 \\ 0 & \delta_{[m}^{[p} \Lambda_{n]}{}^{q]}(x) \end{pmatrix}. \quad (4.9)$$

The undeformed infinitesimal $E_{7(7)}$ symmetry acts on classical vector doublets in the real basis as follows [9]

$$\Delta \begin{pmatrix} F \\ G^0 \end{pmatrix} = \begin{pmatrix} \text{Re}\Lambda - \text{Re}\Sigma & \text{Im}\Lambda + \text{Im}\Sigma \\ -\text{Im}\Lambda + \text{Im}\Sigma & \text{Re}\Lambda + \text{Re}\Sigma \end{pmatrix} \begin{pmatrix} F \\ G^0 \end{pmatrix}. \quad (4.10)$$

The $SU(8)$ doublet is now constructed from the $E_{7(7)}$ doublet and the vielbein as shown in (4.5). The field strength \mathcal{F}_{ij}^0 ($\bar{\mathcal{F}}^{0ij}$) transforms as a **28** (**28**) under $SU(8)$ and it is invariant under $E_{7(7)}$:

$$(\mathcal{F}_{ij}^0, \bar{\mathcal{F}}^{0ij})' = (\mathcal{X}_{IJ}^0, \bar{\mathcal{X}}^{0IJ})' \mathcal{V}' = (\mathcal{X}_{IJ}^0, \bar{\mathcal{X}}^{0IJ}) E E^{-1} \mathcal{V} U(x) = (\mathcal{F}_{ij}^0, \bar{\mathcal{F}}^{0ij}) U(x) . \quad (4.11)$$

The field strength \mathcal{F}_{ij} transforms as a **28** under $SU(8)$ and is space-time complex self-dual

$$\mathcal{F}_{ij\mu\nu}^0 = \frac{1}{2} (\sigma_{\mu\nu})^{\alpha\beta} M_{\alpha\beta ij} , \quad \mathcal{F}_{ij\mu\nu}^{*0} = i \mathcal{F}_{ij\mu\nu}^0 . \quad (4.12)$$

Counterterms depend on vector fields only via $M_{\alpha\beta ij}$ and its conjugate. Equation (4.5) together with the constraint (4.12) allow to find the relation between F and G^0 . It is the same relation which follows from the classical action and definition $G^0 = 2 \frac{\delta S_0}{\delta F}$. One finds that in absence of fermions

$$G_{\dot{\alpha}\dot{\beta}}^0 = -i F_{\dot{\alpha}\dot{\beta}} (U - V)(U + V)^{-1} . \quad (4.13)$$

This relation is the same as the one in (2.4) for $\mathcal{N} = 8$ supergravity.

2. *Scalars*: the scalars enter in counterterms only via a combination which transform under $SU(8)$ but is neutral under $E_{7(7)}$. Namely, the $SU(8)$ covariant derivative of the vielbein is contracted with the inverse vielbein

$$\mathcal{V}^{-1} D \mathcal{V} = \mathcal{V}^{-1} d \mathcal{V} + \begin{pmatrix} Q & 0 \\ 0 & \bar{Q} \end{pmatrix} = \begin{pmatrix} 0 & \bar{P} \\ P & 0 \end{pmatrix} . \quad (4.14)$$

The counterterms depend on $SU(8)$ tensors P_{ijkl} and \bar{P}^{ijkl} , and $\bar{P}^{ijkl} = \frac{1}{4!} \epsilon^{ijklmnpq} P_{mnpq}$. When \mathcal{V} transforms under $E_{7(7)}$ and $SU(8)$ as $\mathcal{V}' = E^{-1} \mathcal{V} U(x)$ the scalar combination in $\mathcal{V}^{-1} D \mathcal{V}$ is $SU(8)$ covariant and $E_{7(7)}$ invariant

$$\mathcal{V}'^{-1} D' \mathcal{V}' = U(x)^{-1} \mathcal{V}^{-1} D \mathcal{V} U(x) . \quad (4.15)$$

The linearized version of $P_{\alpha\dot{\beta}ijkl}$ is a derivative of the scalar field, $P_{\alpha\dot{\beta}ijkl} = \partial_{\alpha\dot{\beta}} \phi_{ijkl} + \dots$ and $\bar{P}_{\alpha\dot{\beta}}^{ijkl} = \partial_{\alpha\dot{\beta}} \bar{\phi}^{ijkl} + \dots$

The torsion and curvature superspace tensors depend on $M_{\alpha\beta ij}$ and its $SU(8)$ conjugate $\bar{M}^{\alpha\beta ij}$ and on $P_{\alpha\dot{\beta}ijkl}$, $\bar{P}_{\alpha\dot{\beta}}^{ijkl}$ and their $SU(8)$ supercovariant derivatives. The counterterms are invariant under undeformed $E_{7(7)}$ symmetry since they depend only on $SU(8)$ tensors constructed from F and $G^0 = 2 \frac{\delta S_0}{\delta F}$. When the action is deformed by the counterterms, the dual field strength is modified

$$G^0 \rightarrow G^0 + \hat{G} . \quad (4.16)$$

Consistency of the deformed duality symmetry requires that $\frac{\delta}{\delta F^\Lambda} \int \left(\hat{G} B \hat{G} \right)$ vanish. Here 70 parameters B are given by $(\text{Im}\Lambda + \text{Im}\Sigma)$.

4.1 \hat{G} from counterterms and $E_{7(7)}$ identity

The counterterms in $\mathcal{N} = 8$ supergravity [3, 4] depend on $F_{\mu\nu}$, $D_\lambda F_{\mu\nu}$, $D_\delta D_\lambda F_{\mu\nu}$ etc. For example, the linearized 3-loop counterterm has terms linear, quadratic and quartic in F and its derivatives. The 3-loop linearized candidate counterterm was presented in [4] as an integral over 16 Grassmann variables. The corresponding superaction is

$$S^{3loop} \sim \kappa^4 \int d^4x d^{16}\theta_B W_{1234}^4(x, \theta) \sim \kappa^4 \int d^4x \left(R^4 + (\partial F)^2 R^2 + \dots \right). \quad (4.17)$$

To get all component expressions one has to perform 16 θ integration. Some of these terms were identified in [11] using helicity formalism of the amplitudes. More recently the candidate 3-loop counterterm was presented in [19] in components, with *51 explicit quartic monomials* depending on all component fields of the theory. Of these *29 depend F and its derivatives*. The 7-loop candidate counterterm has analogous dependence on F , just more derivatives. At the linear level in momentum space it has an extra factor $(s^2 + t^2 + u^2)^2$ in terms of Mandelstam variables

$$S^{7loop} \sim \kappa^{12} \int d^{16}\theta_B \prod_{i=1}^4 d^4p_i W_{1234}(p_i, \theta) (s^2 + t^2 + u^2)^2. \quad (4.18)$$

If the computation in [7] would not show that the theory is UV finite in 3-loops, we would have to add S^{3loop} to the classical action so that the UV divergence can be absorbed. To check the status of NGZ identity for the action $S_0 + S^{3loop}$ we pick up one (from 51) term in eq. (6.8) in [19]. We focus on $(\partial F)^2 R^2$ term which in 2-component notation is given by

$$S_{(\partial F)^2 R^2}^{3loop} \sim x R_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} F^{\dot{\alpha}\dot{\beta}ij} \partial^{\dot{\gamma}\gamma} \partial^{\dot{\delta}\delta} F_{ij}^{\alpha\beta} R_{\alpha\beta\gamma\delta}. \quad (4.19)$$

Here $F^{\dot{\alpha}\dot{\beta}ij}$ and its conjugate $F_{ij}^{\alpha\beta}$ may be viewed as linearized $SU(8)$ tensors which are neutral under classical $E_{7(7)}$ duality symmetry. They are related to $E_{7(7)}$ doublet at the non-linear level, as shown in (4.5). In the linear approximation $F_{ij}^{\alpha\beta} \approx F_{IJ}^{\alpha\beta}$ and $F^{\dot{\alpha}\dot{\beta}ij} \approx F^{\dot{\alpha}\dot{\beta}IJ}$. This means that the deformation of the dual field strength caused by the $(\partial F)^2 R^2$ part of the counterterm is

$$\hat{G}_{\dot{\alpha}\dot{\beta}IJ} \sim \frac{\delta S_{(\partial F)^2 R^2}^{3loop}}{\delta F_{\dot{\alpha}\dot{\beta}}^{IJ}} = x R_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} \partial^{\dot{\gamma}\gamma} \partial^{\dot{\delta}\delta} F_{IJ}^{\alpha\beta} R_{\alpha\beta\gamma\delta}. \quad (4.20)$$

Now we have to test the $E_{7(7)}$ identity in the form (3.15) which in 2-component notation requires that

$$\frac{\delta}{\delta F_{\alpha\beta}^{MN}} \int \left(\hat{G}_{\dot{\alpha}\dot{\beta}IJ} B^{IJKL} \hat{G}_{KL}^{\dot{\alpha}\dot{\beta}} - h.c. \right) = 0. \quad (4.21)$$

Substituting (4.20) into (6.2) we find that

$$x^2 R_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} \partial^{\dot{\gamma}\gamma} \partial^{\dot{\delta}\delta} \left(R_{\alpha\beta\gamma\delta} B^{IJKL} \epsilon^{\dot{\alpha}\dot{\alpha}_1} \epsilon^{\dot{\beta}\dot{\beta}_1} R_{\dot{\alpha}_1\dot{\beta}_1\dot{\gamma}_1\dot{\delta}_1} \partial^{\dot{\gamma}_1\gamma_1} \partial^{\dot{\delta}_1\delta_1} F_{KL}^{\alpha_1\beta_1} R_{\alpha_1\beta_1\gamma_1\delta_1} \right) = 0. \quad (4.22)$$

For generic curvatures and vectors (even with the account of the linearized equations of motion $\partial^{\alpha\dot{\alpha}}R_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} = 0$ and $\partial^{\alpha\dot{\alpha}}F_{\dot{\alpha}\dot{\beta}} = 0$) the $E_{7(7)}$ identity (4.21) is violated unless $x = 0$.

There is no other term in 51 structures of the 3-loop counterterm in [19] which can cancel $R^2\partial^2R^2\partial^2F$ term in (4.21), as can be seen by a direct inspection. The computations of the 3-loop UV divergence in [7] have shown that $x = 0$. Equation (4.22) is an $E_{7(7)}$ symmetry prediction that x has to vanish.

We could have focused on $(\partial F)^4$ term in the 3-loop counterterm, there are two such terms:

$$S_{(\partial F)^4}^{3loop} \sim x \left(F_{\dot{\alpha}\dot{\beta}}^{ij} \partial_\mu \partial_\nu F_{kl}^{\dot{\alpha}\dot{\beta}} \partial^\mu \partial^\nu F_{ij}^{\alpha\beta} F_{\alpha\beta kl} + F_{\dot{\alpha}\dot{\beta}}^{ij} \partial_\mu \partial_\nu F_{kl}^{\dot{\alpha}\dot{\beta}} \partial^\mu F_{ik}^{\alpha\beta} \partial^\nu F_{\alpha\beta jl} \right). \quad (4.23)$$

The corresponding deformation of $\hat{G}_{\dot{\alpha}\dot{\beta}}$ due to these two terms would have 2 terms, from each term in (4.23). In the identity (4.21) there will be $2 \times 2 + 2 \times 2 = 8$ terms since the second term in (4.21) will also contribute. One would have to prove that all 8 terms do not cancel and this would require significantly more computations, as well as the use of various identities. It is therefore nice that in the sector of 2 gravitons and two vectors we find only one contribution to the identity (4.21) presented in (4.22). This expression is not vanishing unless $x = 0$, i.e. the presence of the counterterm would break the $E_{7(7)}$ current conservation in the deformed theory.

We may now look at any exact L -loop counterterm in [3, 4] or in [13] for the linearized form of these counterterms. At the level of a 4-point amplitude we will find terms like $\kappa^{2(L-1)}(\partial F)^2\partial^{2(L-3)}R^2$. For example, the 7-loop counterterm will have $\kappa^{12}(\partial F)^2R^2(s^2 + t^2 + u^2)^2$ terms as well as many other ones. The procedure of getting all required structures for the linearized partners of $D^{2k}R^4$ is described in [19].

The computation of the deformation of the dual field strength to get an explicit expression for \hat{G} becomes more involved since the number of terms with various distribution of extra derivatives grows and there will be more than one term to look at. However, there will be also growing number of structures in (6.2) which all have to vanish. We do not see any possibility to satisfy the identity (6.2) in each sector, unless the coefficient in front of each counterterm vanishes.

5 Deformation of $\mathcal{N} = 8$ supergravity

In this section we would like to find an alternative reason for the $E_{7(7)}$ current conservation forbidding counterterms constructed in [3, 4], which are compatible with all gauge symmetries and are invariant under undeformed $E_{7(7)}$. We consider a possibility to deform the classical action of $\mathcal{N} = 8$ supergravity by the counterterms, which means that the deformed action has to provide a 2-form doublet of $E_{7(7)}$ such that the symmetry mixes vector equations with Bianchi identities:

$$d(\mathcal{X}_{IJ}, \bar{\mathcal{X}}^{IJ}) = 0. \quad (5.1)$$

This means that on shell there are 28 complex (56 real) potentials, $\mathcal{X}_{IJ} = d(\mathcal{B}_{IJ} + i\mathcal{A}^{IJ})$, $\bar{\mathcal{X}}^{IJ} = d(\mathcal{B}_{IJ} - i\mathcal{A}^{IJ})$. The corresponding double set of field strengths is given by

$$F_{\mu\nu}^{IJ} = \partial_\mu \mathcal{A}_\nu^{IJ} - \partial_\nu \mathcal{A}_\mu^{IJ}, \quad G_{\mu\nu IJ} = \partial_\mu \mathcal{B}_{\nu IJ} - \partial_\nu \mathcal{B}_{\mu IJ}. \quad (5.2)$$

Only one of them shows up in the deformed action, the other one must be a functional of the first one, or vice versa, since there are only 28 real vectors in $\mathcal{N} = 8$ supergravity. So we need to find a relation between \mathcal{X}_{IJ} and $\bar{\mathcal{X}}^{IJ}$ (or F^{IJ} and G_{IJ}) which picks up 28 dynamical degrees of freedom out of 56. The corresponding constraint was discovered by Cremmer and Julia in the context of the classical $\mathcal{N} = 8$ supergravity [2]. It requires as the first stage a construction of the $SU(8)$ doublet.

To construct the $SU(8)$ tensors one has to use the vielbein (4.6). Note that the relation between the $SU(8)$ doublet $(\mathcal{F}_{ij}, \bar{\mathcal{F}}^{ij})$ and the $E_{7(7)}$ doublet $(\mathcal{X}_{IJ}, \bar{\mathcal{X}}^{IJ})$ is unique

$$(\mathcal{F}_{ij}, \bar{\mathcal{F}}^{ij}) = (\mathcal{X}_{IJ}, \bar{\mathcal{X}}^{IJ})\mathcal{V}, \quad (5.3)$$

since there is only one scalar dependent object, vielbein, which transforms as $\mathcal{V}' = E^{-1}\mathcal{V}U(x)$ and makes a bridge between the local $SU(8)$ and global $E_{7(7)}$. The $E_{7(7)}$ symmetry acting on scalars is not deformed, whereas the $E_{7(7)}$ symmetry of the deformed $E_{7(7)}$ must be $(\mathcal{X}_{IJ}, \bar{\mathcal{X}}^{IJ})' = (\mathcal{X}_{IJ}, \bar{\mathcal{X}}^{IJ})E$. A simple analogy is the relation between a tangent space vector $V^a = V^\mu e_\mu^a$ and a curved space vector V^μ : V^a is invariant under general coordinate transformations but transforms under Lorentz ones. V^μ is invariant under Lorentz transformations but transforms under general coordinate transformations ones. The vielbein e_μ^a bridges a linear relation between V^a and V^μ .

The unique $SU(8)$ and Lorentz covariant, and $E_{7(7)}$ invariant Cremmer-Julia constraint which reduces the number of real vectors to 28 is

$$\boxed{\mathcal{F}_{ij\mu\nu} + i\mathcal{F}_{ij}^*{}_{\mu\nu} = 0.} \quad (5.4)$$

In this form it corresponds to eq. (16) in [5] and it can also be presented in spinor notation as

$$\mathcal{F}_{ij\mu\nu} = \frac{1}{2}(\sigma_{\mu\nu})^{\alpha\beta}\mathcal{M}_{\alpha\beta ij}, \quad \bar{\mathcal{F}}_{\mu\nu}^{ij} = \frac{1}{2}(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}\dot{\beta}}\bar{\mathcal{M}}_{\dot{\alpha}\dot{\beta}}^{ij}. \quad (5.5)$$

Thus, the field strength \mathcal{F}_{ij} transforms as a **28** under $SU(8)$ and is space-time complex self-dual. The complex conjugate field strength $\bar{\mathcal{F}}^{ij}$ transforms as a **28** under $SU(8)$ and is space-time complex anti-self-dual. One can present it as follows

$$\begin{aligned} \mathcal{F}_{ij\alpha\beta} &= \mathcal{M}_{\alpha\beta ij}, & \mathcal{F}_{ij\dot{\alpha}\dot{\beta}} &= 0, \\ \bar{\mathcal{F}}_{\alpha\beta}^{ij} &= 0, & \bar{\mathcal{F}}_{\dot{\alpha}\dot{\beta}}^{ij} &= \bar{\mathcal{M}}_{\dot{\alpha}\dot{\beta}}^{ij}, \end{aligned} \quad (5.6)$$

and $(\mathcal{M}_{\alpha\beta ij})^\dagger = \bar{\mathcal{M}}_{\dot{\alpha}\dot{\beta}}^{ij}$, $(\bar{\mathcal{F}}_{\alpha\beta}^{ij})^\dagger = \mathcal{F}_{ij\dot{\alpha}\dot{\beta}} = 0$. To see how the $SU(8)$ covariant and $E_{7(7)}$ invariant constraint allows to express $G_{\mu\nu IJ}$ as a functional of $F_{\mu\nu}^{IJ}$ and vice versa we need to use the explicit

relation between the $E_{7(7)}$ doublets and the $SU(8)$ tensors. From eq. (5.3) and (5.6) it follows that $\mathcal{F}_{ij\dot{\alpha}\dot{\beta}} = [(G_{\dot{\alpha}\dot{\beta}} + iF_{\dot{\alpha}\dot{\beta}})U + (G_{\dot{\alpha}\dot{\beta}} - iF_{\dot{\alpha}\dot{\beta}})V]_{ij} = 0$ and therefore, in absence of fermions

$$[G_{\dot{\alpha}\dot{\beta}}(U + V) + iF_{\dot{\alpha}\dot{\beta}}(U - V)]_{ij} = 0. \quad (5.7)$$

The constraint (5.4) has a unique solution for G in terms of F , and vice versa, since the $U + V$ and $U - V$ scalar-dependent matrices are invertible

$$G_{\dot{\alpha}\dot{\beta}} = -iF_{\dot{\alpha}\dot{\beta}}(U - V)(U + V)^{-1}, \quad F_{\dot{\alpha}\dot{\beta}} = iG_{\dot{\alpha}\dot{\beta}}(U + V)(U - V)^{-1}. \quad (5.8)$$

Thus G is a linear function of F : it follows from the unique $SU(8)$ covariant and $E_{7(7)}$ invariant constraint (5.4) on 56 $SU(8)$ field strengths. The linear nature of this relation originates in the linear relation between tangent space $SU(8)$ vectors and curved space $E_{7(7)}$ vectors: they are bridged by the vielbein \mathcal{V} in (5.3) and no other relation is possible.

Counterterms independently of details, would violate this requirement, since higher powers of F will be present in G , so $E_{7(7)}$ current conservation forbids them.

6 Discussion

In conclusion, using either the Noether-Gaillard-Zumino $E_{7(7)}$ current conservation or the uniqueness of the Cremmer-Julia constraint we argued that all candidate counterterms of $\mathcal{N} = 8$ supergravity are forbidden. One may wonder whether it is possible to deform the counterterms in (4.1) so that the deformed duality symmetry is respected. The answer is negative: all non-linear counterterms described in [3,4] provide the unbroken general covariance, local Lorentz and local $SU(8)$ symmetry as well as local supersymmetry. All this is guaranteed by the fact that in classical theory there is an on shell super-geometry and all torsion and curvature forms satisfy the superspace Bianchi identities [5]. Therefore to preserve the undeformed gauge symmetry of the counterterms we have to use the candidate counterterms constructed in [3,4]. These counterterms are invariant under the undeformed $E_{7(7)}$ symmetry. However, they are in conflict with $E_{7(7)}$ current conservation and the duality symmetry of the deformed equations of motion.

The deep reason why it was possible to construct an infinite number of candidate counterterms in [3,4] is the fact that the supersymmetric “tensor calculus” for gauge symmetries of the theory is available [5]. It is the existence of an $\mathcal{N} = 8$ supersymmetric analog (4.1) of the pure gravity case where there is a tangent space Riemann-Christoffel curvature tensor R_{abcd} which allows to construct any higher derivative scalars by contracting any number of curvature tensors with any number of Lorentz covariant derivatives D_a using Minkowski metric η^{ab} so that the number of counterterms proliferates with increasing loop order.

In this paper we have studied hidden $E_{7(7)}$ duality symmetry between 28-component Bianchi identity, $\partial_\mu \tilde{F}^{\mu\nu IJ} = 0$, and 28-component equations of motion of the theory, $\partial_\mu \tilde{G}^{\mu\nu}_{IJ} = 0$. Here $\tilde{G}_{IJ} = 2 \frac{\delta(S_0 + S_{CT})}{\delta F^{IJ}} = \tilde{G}_{0IJ} + \hat{\tilde{G}}_{IJ}$, therefore equations of motion as well as $E_{7(7)}$ duality are deformed by counterterms. The deformed $E_{7(7)}$ duality is

$$\Delta \partial_\mu \tilde{F}^{\mu\nu IJ} = A \partial_\mu \tilde{F}^{\mu\nu IJ} + B \partial_\mu \tilde{G}^{\mu\nu IJ} . \quad (6.1)$$

Here $A = \text{Re}\Lambda - \text{Re}\Sigma$, $B = \text{Im}\Lambda + \text{Im}\Sigma$ are 133 $E_{7(7)}$ symmetry parameters which mix the Bianchi identity with deformed equations of motion. The consistency of the deformed duality requires an extra infinite number of cancellations for the current conservation in the form of an identity (3.15), imposed on the deformation of the dual field strength by the counterterms (4.1). The identity requires that

$$\frac{\delta}{\delta F^{IJ}} \int \hat{\tilde{G}} B \hat{\tilde{G}} = 0 . \quad (6.2)$$

Here $\hat{\tilde{G}}$ is the deformation of the dual field strength caused by counterterms (4.1) where $\hat{\tilde{G}}^{\mu\nu IJ} = 2 \frac{\delta S_{CT}}{\delta F^{IJ}_{\mu\nu}}$. There is no reason for this infinite number of cancellations, as shown in the paper. As an example, for the sector $R^2 \partial^2 R^2 \partial^2 F$ of the identity (3.15) for the 3-loop counterterm the details are given in eq. (4.22). Based on the analysis of the $E_{7(7)}$ current conservation in the form (6.2) we conclude that the hidden $E_{7(7)}$ invalidates all gauge symmetry invariant candidate counterterms.

We also presented a unitarity based argument, independent of specific form of $E_{7(7)}$ current conservation (6.2). It requires to use a simple property of all counterterms: they have terms which are at least quartic in F .

This argument relies on the uniqueness of the constraint which allows to express the 28-component $E_{7(7)}$ complex doublet $(\mathcal{X}_{IJ}, \overline{\mathcal{X}}^{IJ})$ via 28 independent real vector fields of the theory. The $SU(8)$ doublet $(\mathcal{F}_{ij}, \overline{\mathcal{F}}^{ij})$ and the $E_{7(7)}$ doublet have a unique relation via the 133-component vielbein \mathcal{V} , which is

$$(\mathcal{F}_{ij}, \overline{\mathcal{F}}^{ij}) = (\mathcal{X}_{IJ}, \overline{\mathcal{X}}^{IJ}) \mathcal{V} . \quad (6.3)$$

The unitarity constraint is a unique $E_{7(7)}$ invariant, Lorentz and $SU(8)$ covariant constraint and it can be presented in the form $\mathcal{F}_{ij\alpha\beta} = \mathcal{M}_{\alpha\beta ij}, \overline{\mathcal{F}}^{ij}_{\dot{\alpha}\dot{\beta}} = \overline{\mathcal{M}}^{ij}_{\dot{\alpha}\dot{\beta}} (\mathcal{M}_{\alpha\beta ij})^\dagger, \mathcal{F}_{ij\dot{\alpha}\dot{\beta}} = \overline{\mathcal{F}}^{ij}_{\alpha\beta} = 0$, see the derivation in (5.6). This constraint is not valid when counterterms with higher powers of F are added to the action, since it leads to a non-linear relation between $SU(8)$ doublet and $E_{7(7)}$ complex doublet, in contradiction with the $\frac{E_{7(7)}}{SU(8)}$ coset space geometry relation (6.3). The details can be found in Sec. 5.

Thus, in the absence of anomalies perturbative $\mathcal{N} = 8$ supergravity is predicted to be UV finite, in agreement with the light-cone superspace prediction [1].

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A Derivation of the NGZ identity [15, 16]

Note that

$$S[F', \varphi] - S[F, \varphi] = \int \Delta F \frac{\delta S}{\delta F} = \frac{1}{2} \int \Delta F \tilde{G} . \quad (\text{A.1})$$

It is also equal to $S[F', \varphi'] - S[F, \varphi']$ since we are making infinitesimal transformations. It follows that

$$\frac{\delta}{\delta F^\Lambda} \left(S[F', \varphi'] - S[F, \varphi'] \right) = \frac{1}{2} \frac{\delta \Delta F}{\delta F^\Lambda} \tilde{G} + \frac{1}{2} \Delta F \frac{\delta \tilde{G}}{\delta F^\Lambda} . \quad (\text{A.2})$$

We also compute

$$\frac{\delta}{\delta F^\Lambda} \left(S[F, \varphi'] - S[F, \varphi] \right) = \Delta \varphi \frac{\delta^2 S}{\delta F^\Lambda \delta \varphi} = \frac{1}{2} \Delta \varphi \frac{\delta \tilde{G}}{\delta \varphi} . \quad (\text{A.3})$$

Now we sum (A.2) and (A.3) and we get

$$\frac{\delta}{\delta F^\Lambda} \left(S[F', \varphi'] - S[F, \varphi] \right) = \frac{1}{2} \Delta \varphi \frac{\delta \tilde{G}}{\delta \varphi} + \frac{1}{2} \Delta F \frac{\delta \tilde{G}}{\delta F^\Lambda} + \frac{1}{2} \frac{\delta \Delta F}{\delta F^\Lambda} \tilde{G} . \quad (\text{A.4})$$

Now we use the following:

$$\Delta \varphi \frac{\delta \tilde{G}}{\delta \varphi} + \Delta F \frac{\delta \tilde{G}}{\delta F^\Lambda} = \Delta \tilde{G} = C \tilde{F} + D \tilde{G} \quad (\text{A.5})$$

since

$$\Delta \begin{pmatrix} \tilde{F} \\ \tilde{G} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \tilde{F} \\ \tilde{G} \end{pmatrix} . \quad (\text{A.6})$$

We may now continue with eq. (A.4)

$$\frac{\delta}{\delta F^\Lambda} \left(S[F', \varphi'] - S[F, \varphi] \right) = \frac{1}{2} (C \tilde{F} + D \tilde{G}) + \frac{1}{2} (A^T \tilde{G} + \frac{\delta G}{\delta F} B \tilde{G}) = \frac{1}{2} (C \tilde{F} + \frac{\delta G}{\delta F} B \tilde{G}) . \quad (\text{A.7})$$

Here we took into account the properties of the $Sp(2n, \mathbb{R})$ transformations

$$A^\top = -D , \quad B^\top = B , \quad C^\top = C . \quad (\text{A.8})$$

Note that

$$\frac{1}{2} (C \tilde{F} + \frac{\delta G}{\delta F} B \tilde{G}) = \frac{1}{4} \frac{\delta}{\delta F^\Lambda} (F C \tilde{F} + G B \tilde{G}) , \quad (\text{A.9})$$

which proves that

$$\frac{\delta}{\delta F^\Lambda} \left(S[F', \varphi'] - S[F, \varphi] - \frac{1}{4} (F C \tilde{F} + G B \tilde{G}) \right) = 0 . \quad (\text{A.10})$$

In the $SU(8)$ local version of the theory the Noether current of scalars has an elegant form which follows from the action (in absence of fermions)

$$L_{\mathcal{V}} = -\frac{1}{2}\text{Tr}\left((D_{\mu}\mathcal{V})\mathcal{V}^{-1}(D^{\mu}\mathcal{V})\mathcal{V}^{-1}\right). \quad (\text{A.11})$$

$$j_{\mathcal{V}}^{\mu} = \text{Tr}\left(\frac{\partial L}{\partial \mathcal{V}_{,\mu}}\mathcal{V}\delta E^{-1}\right) = -\text{Tr}\left(\mathcal{V}^{-1}(D^{\mu}\mathcal{V})\delta E^{-1}\right). \quad (\text{A.12})$$

We may now introduce the Gaillard-Zumino current [15]

$$\hat{J}^{\mu} \equiv \frac{1}{2}\left(\tilde{G}^{\mu\nu}A\mathcal{A}_{\nu} - \tilde{F}^{\mu\nu}C\mathcal{A}_{\nu} + \tilde{G}^{\mu\nu}B\mathcal{B}_{\nu} - \tilde{F}^{\mu\nu}D\mathcal{B}_{\nu}\right), \quad (\text{A.13})$$

whose divergence cancels the scalar variation of the Lagrangian when equations of motion are satisfied. The classical Lagrangian provides the conservation of the total current, the Noether current of the scalars and the Gaillard-Zumino current of vectors: $\partial_{\mu}J^{\mu} = \partial_{\mu}\hat{J}^{\mu} + \partial_{\mu}j_{\mathcal{V}}^{\mu} = 0$.

B Counterterms and the Hamiltonian approach to duality

In the Hamiltonian approach⁸ to duality symmetries [18], for example in the Coulomb gauge, it is important that in the classical action A_0 is a Lagrange multiplier

$$L = \pi^i \dot{A}_i - H(\pi^i, A_i) + A_0 \partial_i \pi^i, \quad (\text{B.1})$$

and therefore the momenta π^i conjugate to the vector A_i satisfies the constraint $\partial_i \pi^i = 0$. The resolution of this constraint requires to introduce in addition to A_i the second vector potential Z_i

$$\pi^i = -\frac{1}{2}\epsilon^{ijk}(\partial_j Z_k - \partial_k Z_j). \quad (\text{B.2})$$

The fact that the Hamiltonian constraint $\partial_i \pi^i = 0$ is the Gauss' law is fundamental in duality symmetric theories. It explains the doublet nature of potentials in ungauged supergravity theories (A_i, Z_i) , where scalars and fermions interact with vectors only via $F_{\mu\nu}$.

The counterterms in $\mathcal{N} = 8$ supergravity [3, 4] depend on $F_{\mu\nu}$, $D_{\lambda}F_{\mu\nu}$, $D_{\delta}D_{\lambda}F_{\mu\nu}$ etc. If any of such counterterms were added to the classical action, A_0 would not be a Lagrange multiplier anymore. It is not clear *a priori* if the Gauss law and second vector potential required for duality are available. In presence of deformation of the classical action by candidate counterterms with derivatives of $F_{\mu\nu}$ the analysis of dualities in [17], [18] needs to be revisited to find out how the deformations affect duality symmetry of the Hamiltonian/Lorentz non-covariant action. Such analysis will lead to an independent statement about the implications of the continuous global non-compact $E_{7(7)}$ duality on perturbative $\mathcal{N} = 8$ supergravity.

⁸There is a related issue in [17] where the $E_{7(7)}$ symmetry is realized off shell in the Lorentz non-covariant way.

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